Forced vibrations of a nonhomogeneous string

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Abstract

We prove existence of vibrations of a nonhomogeneous string under a nonlinear time periodic forcing term, in the case the forcing frequency avoids resonances with the vibration modes of the string (non-resonant case). The proof relies on a Nash-Moser iteration scheme.[1]

1 Introduction

In this paper we study forced vibrations of a nonhomogeneous string

$$\begin{cases} \rho(x)u_{tt} - (p(x)u_x)_x = \mu f(x,t,u) \\ u(0,t) = u(\pi,t) = 0 \end{cases} \tag{1}$$

where $\rho(x) > 0$ is the mass per unit length, $p(x) > 0$ is the modulus of elasticity multiplied by the cross-sectional area (see [13] p.291), $\mu > 0$ is a parameter, and the nonlinear forcing term $f(x,t,u)$ is $(2\pi/\omega)$-periodic in time.

Equation (1) is a nonlinear model also for propagation of waves in non-isotropic media describing seismic phenomena, see e.g. [2].

We look for $(2\pi/\omega)$-time periodic solutions $u(x,t)$ of (1).

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This problem has received wide attention starting from the pioneering paper of Rabinowitz [21] dealing with the weakly nonlinear homogeneous string $\rho(x) = p(x) = 1$, μ small, and the forcing frequency $\omega = 1$ which enters in resonance with the proper eigen-frequencies $\omega_j = j \in \mathbb{N}$ of the string. For functions 2π-periodic in time and satisfying spatial Dirichlet boundary conditions, the spectrum $\{l^2 - j^2, l \in \mathbb{Z}, j \geq 1\}$ of the D'Alembertian operator $\partial_{tt} - \partial_{xx}$ possesses the zero eigenvalue with infinite multiplicity (resonance) but the remaining eigenvalues are well separated. The corresponding infinite dimensional bifurcation problem is solved in [21] for nonlinearities f which are monotone in u; see [6] and references therein for non-monotone f.

Subsequently many other results, both of bifurcation and of global nature (i.e. $\mu = 1$), have been obtained, still for rational forcing frequencies $\omega \in \mathbb{Q}$, relying on these good separation properties of the spectrum, see e.g. [22, 23, 12, 26, 4] and references therein.

When the forcing frequency $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is irrational (non-resonant case) the situation is completely different. Indeed the D'Alembertian operator $\omega^2\partial_{tt} - \partial_{xx}$ does not possess the zero eigenvalue but its spectrum $\{\omega^2 l^2 - j^2, l \in \mathbb{Z}, j \geq 1\}$ accumulates to zero for almost every ω. This is a "small divisors problem".

We underline that this "small divisors" phenomenon arises naturally for more realistic model equations like (1) where the density $\rho(x)$ and the modulus of elasticity $p(x)$ are not constant. Indeed in this case the eigen-frequencies ω_j of the string are no more integer numbers, having the asymptotic expansion

$$\omega_j^2 = \frac{j^2}{c^2} + b + O\Big(\frac{1}{j}\Big) \tag{2}$$

with suitable constants c, b depending on ρ, p, see (55).

If $\omega = m/n \in \mathbb{Q}$, good separation properties of the spectrum can been recovered when $p(x) = \rho(x)$ (so $c = 1$) and assuming the extra condition $b \neq 0$, see [3, 24]. Indeed in this case the linear spectrum

$$\omega^2 l^2 - \omega_j^2 = \omega^2 l^2 - j^2 - b + O\Big(\frac{1}{j}\Big)$$

possesses at most finitely many zero eigenvalues and the remaining part of the spectrum is far away from zero. On the other hand, if $b = 0$, the eigenvalues with $(l, j) \in (n, m)\mathbb{Z}$ tend to zero (also in the case $\omega \in \mathbb{Q}$!).

Existence of weak solutions in the non-resonant case was proved by Acquistapace [1] for $\rho = 1$, for weak nonlinearities (i.e. μ small), and for a zero measure set of forcing frequencies ω for which the eigenvalues $\omega^2 l^2 - \omega_j^2$

are far away from zero. These frequencies are essentially the numbers whose continued fraction expansion is bounded, see [25].

For a similar zero measure set of frequencies, McKenna [18] has obtained some result when $\mu = 1$, for $\rho = p = 1$, and $f(t, x, u) = g(u) + h(t, x)$ with g uniformly Lipschitz, via a fixed point argument, see also [5]; see [16, 9] for related results using variational methods.

Existence of classical solutions of (1) for a positive measure set of frequencies was proved by Plotnikov-Youngerman [19] for the homogeneous string $\rho = p = 1$, μ small, and f monotone in u. The monotonicity condition allows to control the first coefficient in the asymptotic expansion of the eigenvalues (as in (2)) of some perturbed linearized operator.

Recently Fokam [17] has proved existence of classical periodic solutions for large frequencies ω in a set of asymptotically full measure, for the homogeneous string $\rho = p = 1$ plus a potential, when $\mu = 1$ and $f = u^3 + h(t, x)$ with h a trigonometric polynomial odd in time and space.

In the present paper we prove existence of classical solutions of the non-homogeneous string (1) for every $\rho(x), p(x) > 0$, for general nonlinear terms $f(x, t, u)$, and for (μ, ω) belonging to a large measure Cantor set B_γ, when the ratio μ/ω is small, see Theorem 1, covering both the case $\mu \to 0$ and the case $\omega \to +\infty$.

In the limit $\mu/\omega \to 0$ the solution we find tends to a static equilibrium $v(x)$ with smaller, zero average oscillations $w(x, t)$ of amplitude $O(\mu/\omega)$, see (9),(10) and figure 1. The nonlinearity f selects such v through the infinite dimensional bifurcation equation (7) which possesses non-degenerate solutions under natural assumptions on f, see hypothesys (V). This problem was not present in [17] thanks to the symmetry assumptions on f.

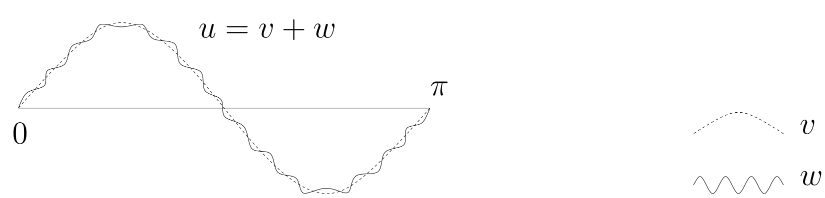

Figure 1: The solution $u(x, t) = v(x) + w(x, t)$ of (1).

Considering the structure of the expected solution it is natural to attack the problem via a Lyapunov-Schmidt decomposition.

In the range equation (to find w) a small divisors problem arises and we solve it with a Nash-Moser type iterative scheme. The inversion of the "linearized operators" — which is the core of any Nash-Moser scheme — is obtained adapting the technique of [7] to the present time-dependent case (section 4). See also [10, 11, 14, 15] for similar techniques.

It is here where the interaction between the forcing frequency ω and the normal modes of oscillations of the string linearized at different positions (approximating better and better the final string configuration) appears. The set B_γ of "non-resonant" parameters (μ, ω) for which we find a solution of the range equation (and then of (1)) is constructed avoiding these primary resonances. In particular the forcing frequency ω must not enter in resonance with the normal frequencies of oscillations of the string linearized at the limiting solution, see (11). At the end of the construction we obtain a large measure Cantor set B_γ which looks like in figure 2. Outside this set the effect of resonance phenomena shall in general destroy the existence of periodic solutions like those found in Theorem 1.

We now present rigorously our results.

1.1 Main result

After a time rescaling we look for 2π-periodic solutions of

$$\begin{cases} \omega^2 \rho(x) u_{tt} - (p(x) u_x)_x = \mu f(x,t,u) \\ u(0,t) = u(\pi,t) = 0 \end{cases} \tag{3}$$

where $\mu \in [0, \bar{\mu}]$ for some $\bar{\mu} > 0$, under the 2π-periodic forcing term

$$f(x,t,u) \;\; = \;\; \sum_{l \in \mathbb{Z}} f_l(x,u) e^{ilt} = f_0(x,u) + \bar{f}(x,t,u) \tag{4}$$

where $\bar{f}(x,t,u) := \sum_{l \neq 0} f_l(x,u) e^{ilt}$.

We suppose that f is analytic in (t,u):

$$f(x,t,u) = \sum_{l \in \mathbb{Z}, k \in \mathbb{N}} f_{lk}(x)\, u^k\, e^{ilt}$$

where $f_{lk}(x) \in H^1((0,\pi);\mathbb{C})$ and[2] $f_{-l,k} = f_{lk}^*$.

Hypothesis (F). *There exist $\sigma_0 > 0$, $r > 0$ such that*

$$\sum_{l \in \mathbb{Z}} \|f_{lk}\|_{H^1}^2 (1 + l^2) e^{(2\sigma_0) 2|l|} := C_k^2(f) < \infty \quad \textit{and} \quad \sum_{k=0}^{+\infty} C_k(f)\, r^k < \infty\,.$$

[2] z^* denotes the complex conjugate of $z \in \mathbb{C}$.

For example, any nonlinearity $f(x,t,u)$ which is a trigonometric polynomial in t and a polynomial in u satisfies hypothesys (F) for every σ_0, r.

If $f(x,t,0) \neq 0$ equation (3) does not possess the trivial solution $u = 0$.

We look for periodic solutions of (3) in the Hilbert space

$$X_{\sigma,s} := \Big\{ u : \mathbb{T} \to H_0^1((0,\pi);\mathbb{R}),\ u(x,t) = \sum_{l\in\mathbb{Z}} u_l(x)\, e^{ilt},\ u_l \in H_0^1((0,\pi);\mathbb{C}),$$
$$u_{-l} = u_l^*,\ \|u\|_{\sigma,s}^2 := \sum_{l\in\mathbb{Z}} \|u_l\|_{H^1}^2 (1 + l^{2s}) e^{2\sigma|l|} < \infty \Big\}$$

of 2π-periodic in time functions valued in $H^1((0,\pi);\mathbb{R})$ which have a bounded analytic extension on the complex strip $|\mathrm{Im}\, t| < \sigma$ with trace function on $|\mathrm{Im}\, t| = \sigma$ belonging to $H^s(\mathbb{T}; H^1((0,\pi);\mathbb{C}))$. For $s > 1/2$, $X_{\sigma,s}$ is an algebra:

$$\|uv\|_{\sigma,s} \le c_s \|u\|_{\sigma,s} \|v\|_{\sigma,s} \quad \forall\, u, v \in X_{\sigma,s}$$

with

$$c_s := 2^s \Big(\sum_{n\in\mathbb{Z}} \frac{1}{1 + n^{2s}} \Big)^{1/2}.$$

We shall use the notation X_σ, resp. $\| \ \|_\sigma$, for $X_{\sigma,1}$, resp. $\| \ \|_{\sigma,1}$.

To find solutions of (3) we implement the Lyapunov-Schmidt reduction according to the decomposition

$$X_{\sigma,s} = V \oplus (W \cap X_{\sigma,s})$$

where

$$V := H_0^1(0,\pi), \qquad W := \Big\{ w = \sum_{l\neq 0} w_l(x)\, e^{ilt} \in X_{0,s} \Big\}$$

writing every $u \in X_{\sigma,s}$ as $u = u_0(x) + \sum_{l\neq 0} u_l(x)\, e^{ilt}$.

Projecting equation (3), with $u(x,t) = v(x) + w(x,t)$, $v \in V$, $w \in W$, yields

$$\begin{cases} -(pv')' = \mu \Pi_V f(v + w) & \text{bifurcation equation} \\ L_\omega w = \mu \Pi_W f(v + w) & \text{range equation} \end{cases} \tag{5}$$

where Π_V, Π_W denote the projectors, $f(u)(x,t) := f(x,t,u(x,t))$ and

$$L_\omega u := \omega^2 \rho(x) u_{tt} - (p(x) u_x)_x\,.$$

We shall find solutions of (5) when the ratio μ/ω is small. In this limit w tends to 0 and the bifurcation equation reduces to the time-independent equation

$$-(pv')' = \mu f_0(v) \tag{6}$$

because, by (4), for $w = 0$,

$$\Pi_V f(v) = \Pi_V f_0(x, v(x)) + \Pi_V \bar{f}(x, t, v(x)) = f_0(v)\,.$$

The infinite dimensional "0-th order bifurcation equation" (6) is a nonautonomous second order ordinary differential equation, which, under natural conditions on f_0, possesses non-degenerate solutions satisfying the boundary conditions $v(0) = v(\pi) = 0$.

Hypothesys (V). *The problem*

$$\begin{cases} -(p(x)v'(x))' = \mu f_0(x, v(x)) \\ v(0) = v(\pi) = 0 \end{cases} \tag{7}$$

admits a solution $\bar{v} \in H_0^1(0, \pi)$ which is non-degenerate, namely the linearized equation

$$-(ph')' = \mu\, f_0'(\bar{v})h$$

possesses in $H_0^1(0, \pi)$ only the trivial solution $h = 0$.

We note that for $\mu = 0$, the trivial solution $\bar{v} = 0$ is non-degenerate, so, by the implicit function theorem, Hypothesis (V) is automatically satisfied for μ small. We deal also with the case μ not small, see for example Lemmas 2 and 3.

For the difficulties with a possibly degenerate solution we refer to [8].

Let λ_j denote the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(p(x)y'(x))' = \lambda \rho(x) y(x) \\ y(0) = y(\pi) = 0 \end{cases} \tag{8}$$

and $\omega_j := \sqrt{\lambda_j}$. These are the frequencies of the free vibrations of the string (note that all the eigenvalues λ_j are positive). Physically, it is the sequence of the fundamental tone ω_1 and all its overharmonics $\omega_2, \omega_3, \ldots$ which compose the musical note of the string.

Theorem 1. *Suppose $p(x), \rho(x) > 0$ are of class $H^3(0, \pi)$, f satisfies (F) and hypothesys (V) holds for some $\mu_0 \in [0, \bar{\mu}]$. Consider the open set*

$$A_0 := \Big\{ (\mu, \omega) \in (\mu_1, \mu_2) \times (\gamma, +\infty) \ : |\omega l - \omega_j| > \frac{\gamma}{l^\tau}\,, \ \forall\, l = 1, \ldots, N_0, \ \ j \geq 1 \Big\}$$

where ω_j are defined by (8), $\gamma \in (0, 1)$, $\tau \in (1, 2)$, (μ_1, μ_2) is a neighborhood of μ_0 (see Lemma 4) and $N_0 = N_0(\rho, p, f, \bar{\mu}, \bar{v}, \tau) \in \mathbb{N}$ is fixed in Lemma 7.

(Existence) *There are constants C, $C' > 0$ depending only on $\rho, p, f, \bar{\mu}, \bar{v}, \tau$, a $\mathcal{C}^\infty$ function*

$$\tilde{w} : \tilde{A} := A_0 \cap \left\{ (\mu, \omega) \ : \ \frac{\mu}{\omega} \le C' \gamma^5 \right\} \to X_{\sigma_0/2} \cap W$$

and a large, see section 3.3, Cantor set $B_\gamma \subset \tilde{A}$, such that for every $(\mu, \omega) \in B_\gamma$ there exists a classical solution of (3)

$$\tilde{u}(\mu, \omega) = v(\mu, \tilde{w}(\mu, \omega)) + \tilde{w}(\mu, \omega) \in V \oplus (W \cap X_{\sigma_0/2}) \tag{9}$$

satisfying

$$\|\tilde{w}(\mu, \omega)\|_{\sigma_0/2} \le C \frac{\mu}{\gamma \omega}, \qquad \|v(\mu, \tilde{w}(\mu, \omega)) - v(\mu, 0)\|_{H^1} \le C \frac{\mu}{\gamma \omega} \tag{10}$$

and $\|v(\mu, 0) - \bar{v}\|_{H^1} \le C|\mu - \mu_0|$. The Cantor set B_γ is explicitely

$$B_\gamma := \Big\{ (\mu, \omega) \in (\mu_1, \mu_2) \times (2\gamma, +\infty) \ : \ |\omega l - \omega_j| > \frac{2\gamma}{l^\tau} \quad \forall\, l = 1, \ldots, N_0, \ j \ge 1,$$
$$\frac{\mu}{\omega} \le C' \gamma^5, \quad \left| \omega l - \frac{j}{c} \right| > \frac{2\gamma}{l^\tau}, \quad |\omega l - \tilde{\omega}_j(\mu, \omega)| > \frac{2\gamma}{l^\tau} \quad \forall\, l, j \ge 1 \Big\} \tag{11}$$

where

$$c := \frac{1}{\pi} \int_0^\pi \left(\frac{\rho(x)}{p(x)} \right)^{1/2} dx \tag{12}$$

and $\tilde{\lambda}_j(\mu, \omega) := \tilde{\omega}_j^2(\mu, \omega)$ denote the (possibly negative[3]) eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(py')' - \mu \Pi_V f'(v(\mu, \tilde{w}(\mu, \omega)) + \tilde{w}(\mu, \omega))\, y = \lambda \rho y \\ y(0) = y(\pi) = 0\,. \end{cases} \tag{13}$$

(Regularity) *Suppose, furthermore, $\rho(x) \in H^m(0, \pi)$, $p(x) \in H^{m+1}(0, \pi)$, $f_{lk}(x) \in H^m(0, \pi)$ and $\sum_{l,k} \|f_{lk}\|_{H^m} r_m^k < \infty$ for some $m \ge 3$, $r_m > 0$. If $\|\tilde{u}(\cdot\,, t)\|_{H_0^1} < r_m / C_m$ for some $C_m > 0$, then $\tilde{u}(\cdot\,, t) \in H_0^1(0, \pi) \cap H^{m+2}(0, \pi)$.*

This conclusion holds true, for example, when $f_0(x, 0) = d_u f_0(x, 0) = 0$ (so $v(\mu, 0) = 0$, $\forall \mu$) for $\mu / \gamma \omega$ small enough, by (10).

Fixed μ, for every frequency ω in the section

$$S(\mu) := \left\{ \omega : (\mu, \omega) \in \cup_{\gamma \in (0,1)} B_\gamma \right\}$$

[3]In this case $\omega_j(\mu, \omega) = i \sqrt{|\lambda_j(\mu, \omega)|}$ is a purely imaginary complex number.

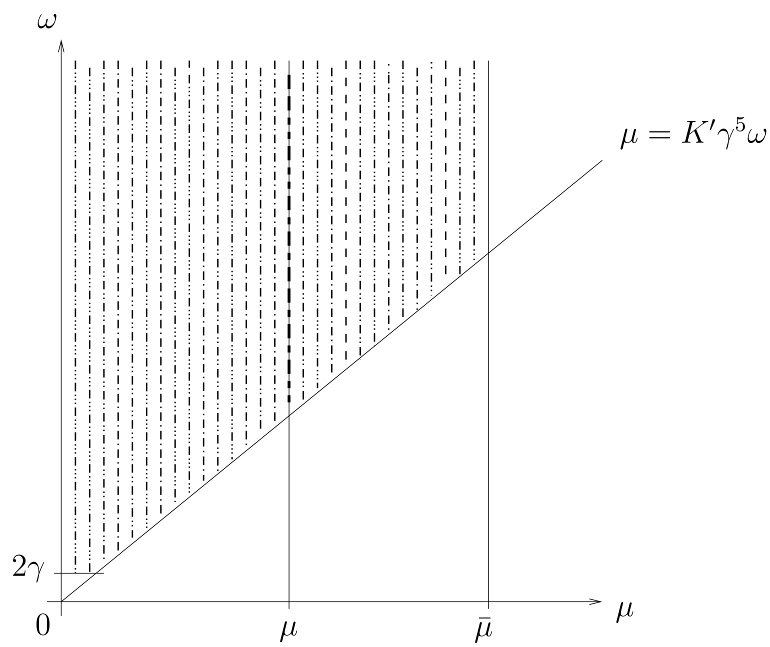

Figure 2: The Cantor set B_γ.

there exists a solution of (1) by Theorem 1. $S(\mu)$ has asymptotically full measure at $\omega \to +\infty$, i.e.

$$\lim_{\omega\to+\infty} |S(\mu) \cap (\omega, \omega+1)| = 1 \, .$$

Analogously, fixed ω, for every μ in the section

$$S(\omega) := \left\{ \mu : (\mu, \omega) \in \cup_{\gamma\in(0,1)} B_\gamma \right\}$$

there exists a solution of (1). For μ small enough, also $S(\omega)$ is a "large" set: for every $\gamma' \in (0,1)$, $\omega' > 0$,

$$\lim_{\mu\to 0} \left| \left\{ \omega \in (\omega', \omega'+1) : \ \frac{|S(\omega) \cap (0,\mu)|}{\mu} \geq 1 - \gamma' \right\} \right| = 1 \, ,$$

see section 3.3.

Notations. The symbols K, K_i shall denote positive constants depending only on $\rho, p, f, \bar{\mu}, \bar{v}, \tau$.

2 The bifurcation equation

We first prove the analyticity of the Nemitski operator induced by f.

Lemma 1. *Let f satisfy assumption (F). For every $\sigma \in [0, \sigma_0]$, $s > 1/2$, the Nemitski operator f is analytic on the ball $\{u \in X_{\sigma,s} : c_s \|u\|_{\sigma,s} < r\}$.*

Proof. First note that

$$\sum_{l \in \mathbb{Z}} \|u_l\|_\infty \leq \sqrt{\frac{\pi}{2}} \sum_{l \in \mathbb{Z}} \|u_l\|_{H^1} \leq \sqrt{\frac{\pi}{2}} \Big(\sum_{l \in \mathbb{Z}} \|u_l\|_{H^1}^2 (1 + l^{2s}) \Big)^{1/2} \Big(\sum_{l \in \mathbb{Z}} \frac{1}{1 + l^{2s}} \Big)^{1/2}$$

so $\|u\|_\infty \leq c_s \|u\|_{\sigma,s}$, $\forall \, u \in X_{\sigma,s}$, $\sigma \geq 0$, $s > 1/2$, and $f(x, t, u(x,t))$ is well-defined.

By definition of the norm $\| \; \|_{\sigma,s}$, there exists $C := C(\sigma_0, s) > 0$ such that $\forall \sigma \in [0, \sigma_0]$, $\forall k \in \mathbb{N}$,

$$\big\| \sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt} \big\|_{\sigma,s} \leq C \, \big\| \sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt} \big\|_{2\sigma_0, 1} = C \, C_k(f) < +\infty$$

by the assumption (F). Hence, for $c_s \|u\|_{\sigma,s} < r$, using the algebra property of $X_{\sigma,s}$,

$$\begin{aligned} \|f(u)\|_{\sigma,s} &\leq \sum_{k=0}^{\infty} \big\| \big(\sum_{l \in \mathbb{Z}} f_{lk}(x) e^{ilt} \big) u^k \big\|_{\sigma,s} \leq C \sum_{k=0}^{\infty} C_k(f) \, (c_s \|u\|_{\sigma,s})^k \\ &< C \sum_{k=0}^{\infty} C_k(f) \, r^k < +\infty \end{aligned}$$

again by (F). The analyticity of the Nemitski operator f w.r.t. $\| \; \|_{\sigma,s}$ follows from the properties of the power series (see e.g. [20], Appendix A). ■

Throughout this paper we shall use the spaces $X_{\sigma,s}$ with $\sigma \in [\sigma_0/2, \sigma_0]$ and $s \in \mathcal{S} := \{1, \, 1 - \frac{\tau - 1}{2}, \, 1 + \frac{(\tau-1)\tau}{2 - \tau}\}$. So we can choose a multiplicative algebra constant on $X_{\sigma,s}$ and a radius R_0 such that in the ball $\{u \in X_{\sigma,s} : \|u\|_{\sigma,s} < R_0\}$ f is analytic, and $f, f', f'', \ldots$ are bounded, uniformly in σ, s.

We now give an example in which hypothesis (V) holds.

Lemma 2. *Suppose $f_0(x, u) = u^m$ for $m \geq 3$ odd and $p(x) \equiv 1$. Then, $\forall \mu$, there exists an unbounded sequence of non-degenerate solutions v_n of (7).*

Proof. All the solutions of the autonomous equation $-v'' = \mu v^m$ are periodic and can be parametrized by their energy

$$E = \frac{1}{2} v'^2 + \frac{\mu}{m+1} v^{m+1} \, .$$

We denote T_E the period of the solution v_E. We can suppose $v_E(0) = 0$, so $v_E'(0) = \sqrt{2E}$. The other boundary condition $v_E(\pi) = 0$ is satisfied iff

$$k \frac{T_E}{2} = \pi \qquad \text{for some } \ k \in \mathbb{N}\,. \tag{14}$$

By symmetry and energy conservation $v_E(T_E/4) = [(m+1)E/\mu]^{\frac{1}{m+1}}$. So

$$\begin{aligned} T_E &= 4 \int_0^{\left[\frac{(m+1)E}{\mu}\right]^{\frac{1}{m+1}}} \left[2\Big(E - \frac{\mu\, x^{m+1}}{m+1} \Big) \right]^{-1/2} dx \\ &= \frac{4(m+1/\mu)^{\frac{1}{m+1}}}{E^{\frac{1}{2}-\frac{1}{m+1}}} \int_0^1 \frac{dy}{\sqrt{2(1-y^{m+1})}} = \frac{C(m,\mu)}{E^{\frac{1}{2}-\frac{1}{m+1}}} \end{aligned}$$

by the change of variable $y = x\,[E(m+1)/\mu]^{-\frac{1}{m+1}}$, and (14) is satisfied at infinitely many energy levels. Let $\bar{E} > 0$ such that $T_{\bar{E}} = 2\pi/k$ and denote the solution $\bar{v} := v_{\bar{E}}$.

Let us prove that $\bar{v}$ is non-degenerate. Any solution h of the linearized equation at $\bar{v}$,

$$-h''(x) = \mu m\, \bar{v}^{\,m-1}(x)\, h(x)\,, \tag{15}$$

can be written as $h = A\bar{v}' + B\beta$, $A, B \in \mathbb{R}$, because $\bar{v}'(x)$ and $\beta(x) := (\partial_E v_E)_{|E=\bar{E}}(x)$ are solutions of (15); they are independent because $\bar{v}'(0) \neq 0$ while $\beta(0) = 0$. If $h(0) = 0$ then $A = 0$. We claim that $\beta(\pi) \neq 0$; as a consequence, if $h(\pi) = 0$, then $B = 0$, and so $h = 0$, i.e. $\bar{v}$ is non-degenerate. To prove that $\beta(\pi) \neq 0$, we differentiate at $\bar{E}$ the identity $v_E(kT_E/2) = 0$,

$$\beta(\pi) + \bar{v}'(\pi)(\partial_E T_E)_{|E=\bar{E}} = 0\,.$$

Since $\bar{v}'(\pi) = (-1)^k \sqrt{2E} \neq 0$ and $\partial_E T_E \neq 0$, we get $\beta(\pi) \neq 0$. ∎

Lemma 3. *If $f_0(x,0) = d_u f_0(x,0) = 0$, then $\bar{v} = 0$ is a non-degenerate solution of* (7) *for every μ.*

Proof. The linearized equation $-(ph')' = 0$, $h(0) = h(\pi) = 0$ has only the trivial solution. ∎

When hypothesys (V) holds at some $(\mu_0, \bar{v})$, we solve first the bifurcation equation in (5) using the standard implicit function theorem. We find, for every w small enough and μ in a neighborhood of μ_0, a unique solution $v(\mu, w)$ of the bifurcation equation.

Lemma 4. *There exist $0 < R < R_0$, a neighborhood $[\mu_1, \mu_2]$ of μ_0 and, $\forall\, \sigma \in [\sigma_0/2, \sigma_0]$, $s \in \mathcal{S}$, a $\mathcal{C}^\infty$ map*

$$[\mu_1, \mu_2] \times \big\{ w \in W \cap X_{\sigma,s} \,:\, \|w\|_{\sigma,s} < R \big\} \to V, \quad (\mu, w) \mapsto v(\mu, w)$$

which solves the bifurcation equation in (5).

Proof. The linear operator

$$h \mapsto -(ph')' - \mu_0 d_v \Pi_V f(v)[h] = -(ph')' - \mu_0\, f_0'(v)\, h$$

is invertible on $H_0^1(0, \pi)$ by hypothesys (V). ■

Remark 1. *The solutions of the* 0*th order bifurcation equation (7) found in Lemmas 2 and 3 are non-degenerate for every μ, so, in such a case, we can continue $v(\mu, w)$ for all $[\mu_1, \mu_2] = [0, \bar{\mu}]$.*

We denote by $\lambda_j(\mu, w) := \omega_j^2(\mu, w)$ the possibly negative eigenvalues of the Sturm-Liouville problem

$$\begin{cases} -(py')' - \mu \Pi_V f'(v(\mu, w) + w)\, y = \lambda \rho y \\ y(0) = y(\pi) = 0\,. \end{cases} \tag{16}$$

By a comparison principle, see the Appendix, the eigenvalues of (16) satisfy

$$|\lambda_j(\mu, w) - \lambda_j(\mu', w')| \le K \Big(|\mu - \mu'| + \|w - w'\|_{\sigma,s} \Big)\,. \tag{17}$$

The non-degeneracy of $\bar{v} := v(\mu_0, 0)$ means that $\lambda_j(\mu_0, 0) \neq 0\ \forall j$ and by (17)

$$\delta_0 := \inf \Big\{ \big|\lambda_j(\mu, w)\big| \,:\, j \ge 1,\ \ \mu \in [\mu_1, \mu_2],\ \ \|w\|_{\sigma_0/2} \le R \Big\} > 0 \tag{18}$$

eventually taking R smaller. Note also that the index j_0 of the smallest positive eigenvalue is constant, independently on (μ, w).

3 Solution of the range equation

It remains to solve the range equation

$$L_\omega w = \mu \Pi_W \mathcal{F}(\mu, w) \tag{19}$$

where

$$\mathcal{F}(\mu, w) := f(v(\mu, w) + w)\,.$$

By the previous lemmas, $\mathcal{F}$ is $\mathcal{C}^\infty$ and bounded, togheter with its derivatives, on $[\mu_1, \mu_2] \times B_R$ where $B_R := \{w \in W \cap X_{\sigma,s} \,:\, \|w\|_{\sigma,s} < R\}$.

3.1 The Nash-Moser recursive scheme

We define the sequence of finite-dimensional subspaces

$$W^{(n)} := \Big\{ w = \sum_{1\le|l|\le N_n} w_l(x)e^{ilt} \Big\} \subset W$$

where $N_n := N_0 2^n$ and $N_0 \in \mathbb{N}$. We also set

$$W^{(n)\perp} := \Big\{ w = \sum_{|l|>N_n} w_l(x)e^{ilt} \in W \Big\}$$

and denote P_n the projection on $W^{(n)}$, $P_n^\perp$ on $W^{(n)\perp}$. For $w \in W^{(n)\perp}$ the following smoothing estimate holds: if $0 < \sigma'' < \sigma'$,

$$\|w\|_{\sigma'',s} \le \exp[-(\sigma' - \sigma'')N_n]\,\|w\|_{\sigma',s}\,. \tag{20}$$

The key property for the construction of the iterative sequence is the invertibility of the linear operator

$$\begin{aligned} \mathcal{L}_n(w)h &:= -L_\omega h + \mu P_n[d_w\mathcal{F}(\mu,w)h] \qquad (21)\\ &= -L_\omega h + \mu P_n\big[f'(v(\mu,w)+w)\big(h + d_w v(\mu,w)[h]\big)\big] \quad \forall\, h \in W^{(n)}\,. \end{aligned}$$

Lemma 5. ***(Inversion of the linear problem)*** *Let $\tau \in (1,2)$, $\gamma \in (0,1)$, $\sigma \in (0,\sigma_0]$. Assume*[4] *$\omega > \gamma$ and the non-resonance conditions:*

$$\Big|\omega l - \frac{j}{c}\Big| > \frac{\gamma}{l^\tau} \qquad \forall\, l = 1,2,\ldots,N_n, \quad \forall\, j \ge 1 \tag{22}$$

where c is defined in (12), and

$$|\omega^2 l^2 - \lambda_j(\mu,w)| > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1,2,\ldots,N_n,\ j \ge 1 \tag{23}$$

where $\lambda_j(\mu,w)$ are the eigenvalues of (16).

Let $u := v(\mu,w) + w$. There exist K_1, K_1' such that, if

$$\frac{\mu}{\gamma^3\omega}\,\|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau(\tau-1)}{2-\tau}} < K_1'\,, \tag{24}$$

then $\mathcal{L}_n(w)$ is invertible and

$$\|\mathcal{L}_n(w)^{-1}h\|_\sigma \le \frac{K_1 N_n^{\tau-1}}{\gamma\omega}\,\|h\|_\sigma \qquad \forall\, h \in W^{(n)}\,. \tag{25}$$

[4] This means that equation (3) is non-autonomous indeed.

Proof. In section 4. ■

For $\vartheta := 3\sigma_0/\pi^2$ we define the sequence

$$\sigma_{n+1} := \sigma_n - \frac{\vartheta}{(n+1)^2}\,, \qquad \sigma_0 > \sigma_1 > \sigma_2 > \ldots > \frac{\sigma_0}{2}\,. \tag{26}$$

Lemma 6. ***(The approximate solution)*** *If $(\mu,\omega) \in A_0$ and $\mu N_0^{\tau-1}/\gamma\omega < K_2'$ is sufficiently small, then there exists a solution $w_0 := w_0(\mu,\omega) \in W^{(0)}$ of*

$$L_\omega w_0 = \mu P_0 \mathcal{F}(\mu, w_0)$$

satisfying $\|w_0\|_{\sigma_0} \le \mu K_2 N_0^{\tau-1}/\gamma\omega$ for some K_2.

Proof. By definition of A_0 in Theorem 1, the eigenvalues of $(1/\rho)L_\omega$ satisfy

$$|\omega^2 l^2 - \lambda_j| > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1,2,\ldots,N_0, \;\; \forall\, j \ge 1\,,$$

so L_ω is invertible on $W^{(0)}$ and, for some K,

$$\|L_\omega^{-1} h\|_{\sigma_0} \le \frac{K N_0^{\tau-1}}{\gamma\omega}\,\|h\|_{\sigma_0} \qquad \forall\, h \in W^{(0)}\,. \tag{27}$$

Then we look for a solution $w_0 \in W^{(0)}$ of $w_0 = \mu L_\omega^{-1} P_0 \mathcal{F}(\mu, w_0)$. The right-hand side term is a contraction in $\{\|w_0\|_{\sigma_0} < R\}$ if $\mu N_0^{\tau-1}/\gamma\omega$ is sufficiently small. ■

Given $w_n \in W^{(n)}$, $\|w_n\|_{\sigma_n} < R$ and $A_n \subseteq A_0$, we define

$$A_{n+1} := \Big\{ (\mu,\omega) \in A_n \,:\, \big|\omega l - \omega_j(\mu, w_n)\big| > \frac{\gamma}{l^\tau}\,, \quad \Big|\omega l - \frac{j}{c}\Big| > \frac{\gamma}{l^\tau}\,, \\ \forall\, l = 1,2,\ldots,N_{n+1}, \;\; j \ge 1 \Big\} \subseteq A_n$$

where $\lambda_j(\mu, w_n) = \omega_j^2(\mu, w_n)$ are defined in (16) with $w = w_n$.

In Lemma 6 we have constructed $h_0 := w_0$ for $(\mu,\omega) \in A_0$. Next, we proceed by induction. By means of w_0 we define the set A_1 as above, and we find $w_1 := h_0 + h_1 \in W^{(1)}$ for every $(\mu,\omega) \in A_1$ by Lemma 7 below. Then we define A_2, we find $w_2 \in W^{(2)}$ and so on. The main goal of the construction is to prove that, at the end of the recurrence, the set of parameters $(\mu,\omega) \in \cap_n A_n$ is actually a large set.

Lemma 7. ***(Inductive step).*** *Fix $\chi \in (1,2)$. Suppose that $h_i \in W^{(i)}$, $\forall i = 0,\ldots,n$, satisfy*

$$\|h_i\|_{\sigma_i} < \frac{\mu K_2 N_0^{\tau-1}}{\gamma\omega}\exp(-\chi^i) \tag{28}$$

where K_2 is the constant in Lemma 6; $\forall k = 0, \dots, n$, $w_k := h_0 + \dots + h_k$ satisfies $\|w_k\|_{\sigma_k} < R$ and

$$L_\omega w_k = \mu P_k \mathcal{F}(\mu, w_k) \tag{29}$$

and suppose that $(\mu, \omega) \in A_n$, where A_{i+1} is constructed by means of w_i as showed above.

There exist $N_0 = N_0(\rho, p, f, \bar{\mu}, \bar{v}, \tau) \in \mathbb{N}$ and K_3' such that: if $(\mu, \omega) \in A_{n+1}$ and $\mu/\gamma^3\omega < K_3'$, then there exists $h_{n+1} \in W^{(n+1)}$ satisfying

$$\|h_{n+1}\|_{\sigma_{n+1}} < \frac{\mu K_2 N_0^{\tau-1}}{\gamma\omega} \exp(-\chi^{n+1}) \tag{30}$$

such that $w_{n+1} = w_n + h_{n+1}$ verifies $\|w_{n+1}\|_{\sigma_{n+1}} < R$ and

$$L_\omega w_{n+1} = \mu P_{n+1} \mathcal{F}(\mu, w_{n+1})\,. \tag{31}$$

Proof. In short $\mathcal{F}(w) := \mathcal{F}(\mu, w)$ and $D\mathcal{F}(w) := d_w\mathcal{F}(\mu, w)$. Equation (31) for $w_{n+1} = w_n + h_{n+1}$ is $L_\omega[w_n + h_{n+1}] = \mu P_{n+1}\mathcal{F}(w_n + h_{n+1})$.

By assumption, w_n satisfies (29) for $k = n$, namely $L_\omega w_n = \mu P_n \mathcal{F}(w_n)$, so the equation for h_{n+1} can be written as

$$\mathcal{L}_{n+1}(w_n)h_{n+1} + \mu(P_{n+1} - P_n)\mathcal{F}(w_n) + \mu P_{n+1} Q = 0 \tag{32}$$

where, as defined in (21), $\mathcal{L}_{n+1}(w_n)h_{n+1} := -L_\omega h_{n+1} + \mu P_{n+1} D\mathcal{F}(w_n)h_{n+1}$, and Q denotes the quadratic remainder

$$Q = Q(w_n, h_{n+1}) := \mathcal{F}(w_{n+1}) - \mathcal{F}(w_n) - D\mathcal{F}(w_n)h_{n+1}\,.$$

Step 1: Inversion of $\mathcal{L}_{n+1}(w_n)$. We verify the assumptions of Lemma 5. By definition of A_{n+1}, ω satisfies (22). If $\lambda_j(\mu, w_n) < 0$, then $|\omega^2 l^2 - \lambda_j(\mu, w_n)| \geq \omega^2 l^2 > \gamma\omega/l^{\tau-1}$ because $\omega > \gamma$. If $\lambda_j(\mu, w_n) > 0$, we have

$$|\omega^2 l^2 - \lambda_j(\mu, w_n)| \geq |\omega l - \omega_j(\mu, w_n)|\omega l > \frac{\gamma\omega}{l^{\tau-1}} \qquad \forall\, l = 1, \dots, N_{n+1}$$

because $(\mu, \omega) \in A_{n+1}$. In both cases the non-resonance condition (23) holds.

To verify (24) we need an estimate for w_n. Let $\eta := \tau(\tau-1)/(2-\tau)$ and $\alpha > 0$. Using the elementary inequality

$$\frac{1 + l^{2(1+\eta)}}{1 + l^2} \cdot \frac{e^{2(\sigma-\alpha)|l|}}{e^{2\sigma|l|}} \leq \frac{2l^{2\eta}}{e^{2\alpha|l|}} \leq 2\Big(\frac{\eta}{\alpha e}\Big)^{2\eta}, \quad \forall\, l \neq 0\,,$$

we deduce

$$\|h_i\|_{\sigma_{n+1}, 1+\eta} \leq \frac{C_\eta}{(\sigma_i - \sigma_{n+1})^\eta} \|h_i\|_{\sigma_i}$$

where $C_\eta := \sqrt{2}(\eta/e)^\eta$. Since $\sigma_i - \sigma_{n+1} \geq \sigma_i - \sigma_{i+1}$ for every $i \leq n$,

$$\|w_n\|_{\sigma_{n+1},1+\eta} \leq \sum_{i=0}^{n} \|h_i\|_{\sigma_{n+1},1+\eta} \leq C_\eta \sum_{i=0}^{n} \frac{\|h_i\|_{\sigma_i}}{(\sigma_i - \sigma_{i+1})^\eta} \leq S_\eta \frac{\mu K_2 N_0^{\tau-1}}{\gamma\omega}$$

using (28) where $S_\eta := (C_\eta/\vartheta^\eta)\sum_{i=0}^{+\infty}(i+1)^{2\eta}\exp(-\chi^i) < +\infty$. If $S_\eta\, \mu K_2 N_0^{\tau-1}/\gamma\omega < R$ then $\|f'(u_n)\|_{\sigma_{n+1},1+\eta} \leq K$ where $u_n := v(\mu, w_n) + w_n$, and hypotheses (24) is verified for $\mu/\gamma^3\omega < K'$ sufficiently small.

Analogously we get $\|w_n\|_{\sigma_n} < R$ if $\mu N_0^{\tau-1}/\gamma\omega < K''$ is small enough.

By Lemma 5 the operator $\mathcal{L}_{n+1}(w_n)$ is invertible on $W^{(n+1)}$ and

$$\|\mathcal{L}_{n+1}(w_n)^{-1}h\|_{\sigma_{n+1}} \leq \frac{K_1 N_{n+1}^{\tau-1}}{\gamma\omega}\|h\|_{\sigma_{n+1}}, \quad \forall h \in W^{(n+1)}. \tag{33}$$

Equation (32) amounts to the fixed point problem

$$h_{n+1} = -\mu\mathcal{L}_{n+1}(w_n)^{-1}\Big[(P_{n+1} - P_n)\mathcal{F}(w_n) + P_{n+1}Q\Big] := \mathcal{G}(h_{n+1})$$

for $h_{n+1} \in W^{(n+1)}$.

Step 2: $\mathcal{G}$ is a contraction. We prove that $\mathcal{G}$ is a contraction on the ball $B_{n+1} := \{\|h\|_{\sigma_{n+1}} < r_{n+1}\}$ where $r_{n+1} := (\mu K_2 N_0^{\tau-1}/\gamma\omega)\exp(-\chi^{n+1})$, implying (30). By (20)

$$\|(P_{n+1} - P_n)\mathcal{F}(w_n)\|_{\sigma_{n+1}} \leq \|\mathcal{F}(w_n)\|_{\sigma_n}\exp[-(\sigma_n - \sigma_{n+1})N_n].$$

Since $\|w_n\|_{\sigma_n} < R$, we have $\|Q\|_{\sigma_{n+1}} \leq K\|h_{n+1}\|^2_{\sigma_{n+1}}$. Hence, by (33),

$$\|\mathcal{G}(h_{n+1})\|_{\sigma_{n+1}} \leq K\frac{\mu N_{n+1}^{\tau-1}}{\gamma\omega}\Big(\exp[-(\sigma_n - \sigma_{n+1})N_n] + \|h_{n+1}\|^2_{\sigma_{n+1}}\Big).$$

Therefore $\mathcal{G}(B_{n+1}) \subseteq B_{n+1}$ if

$$\frac{\mu K N_{n+1}^{\tau-1}}{\gamma\omega}\exp[-(\sigma_n - \sigma_{n+1})N_n] < \frac{r_{n+1}}{2}, \qquad \frac{\mu K N_{n+1}^{\tau-1}}{\gamma\omega} r_{n+1}^2 < \frac{r_{n+1}}{2}. \tag{34}$$

By the definition of σ_n in (26) and $N_n := N_0 2^n$, the first inequality is verified for every $n \geq 0$ if $\sigma_0 N_0$ is greater than a constant depending only on χ, K, K_2. The second inequality is verified for every $n \geq 0$ if $\mu N_0^{\tau-1}/\gamma\omega < K'$ is small enough.

The estimate for $\|\mathcal{G}h - \mathcal{G}k\|$, $h, k \in B_{n+1}$ is similar. By the Contraction Mapping Theorem we conclude. ■

Corollary 1. ***(Existence)*** *Suppose* $A_\infty := \bigcap_{n\geq 0} A_n \neq \emptyset$. *If* $(\mu,\omega) \in A_\infty$ *then*

$$w_\infty(\mu,\omega) := \sum_{n\geq 0} h_n(\mu,\omega) \in W \cap X_{\sigma_0/2}$$

is a solution of the range equation (19) *satisfying* $\|w_\infty\|_{\sigma_0/2} \leq K_3\mu/\gamma\omega$, *and*

$$u_\infty := v(\mu, w_\infty(\mu,\omega)) + w_\infty(\mu,\omega) \in X_{\sigma_0/2}$$

is a classical solution of (3) *satisfying* (10).

Proof. Since w_n solves (29) for $k = n$, $-L_\omega w_n + \mu\Pi_W f(u_n) = \mu P_n^\perp f(u_n) \in W^{(n)\perp}$ where $u_n := v(\mu, w_n) + w_n$. By (20)

$$\lim_{n\to+\infty} \| - L_\omega w_n + \mu f(u_n)\|_{\sigma_0/2} \leq \lim_{n\to+\infty} K \exp[-(\sigma_n - \sigma_0/2)N_n] = 0\,.$$

Since $w_n \to w_\infty$ in $\| \ \|_{\sigma_0/2}$ also $f(u_n) \to f(u_\infty)$ in the same norm, while $L_\omega w_n \to L_\omega w_\infty$ in the sense of distributions. So w_∞ is a weak solution of the range equation (19) and $u_\infty := v(\mu, w_\infty(\mu,\omega))+ w_\infty(\mu,\omega) \in X_{\sigma_0/2}$ is a weak solution of (3).

Finally, by the equation, $\partial_x(p(x)\partial_x u_\infty(x,t))$ is a continuous function in (x,t) and, $\forall t$, $u_\infty(\cdot\,,t) \in H^3(0,\pi) \subset \mathcal{C}^2$ is a classical solution of (3). ■

Remark 2. *We shall prove, as a consequence of Lemma 11 and section 3.3, that* A_∞ *is actually a positive measure set. A possible way to prove it uses the Whitney extension of* w_∞*, see section 3.2.*

Lemma 8. ***(Regularity)*** *Suppose* $\rho(x) \in H^m(0,\pi)$, $p(x) \in H^{m+1}(0,\pi)$, $f_{lk}(x) \in H^m(0,\pi)$ *and* $\sum_{l,k} \|f_{lk}\|_{H^m} r_m^k < \infty$ *for some* $m \geq 3$, $r_m > 0$.

There exists K_m *such that if the solution* u_∞ *of Corollary 1 satisfies* $\|u_\infty(\cdot\,,t)\|_{H^1} < K_m$, *then* $u_\infty(\cdot\,,t) \in H^{m+2}(0,\pi) \cap H_0^1(0,\pi)$.

Proof. For every fixed t, by the algebra property of H^m

$$\big\|f(x,t,u(x,t))\big\|_{H^m} \leq \sum_{l,k} \|f_{lk}(x)u^k(x)\|_{H^m} \leq C \sum_{l,k} \|f_{lk}\|_{H^m} \|u^k\|_{H^m}\,.$$

Using the Gagliardo-Nirenberg type inequality $\|u^k\|_{H^m} \leq (C_m\|u\|_{H^1})^{k-1}\|u\|_{H^m}$ valid for every $u \in H_0^1 \cap H^m$, we get

$$\big\|f(x,t,u(x,t))\big\|_{H^m} \leq C\|u\|_{H^m} \sum_{l,k} \|f_{lk}\|_{H^m} \big(C_m\|u\|_{H^1}\big)^{k-1} \qquad (35)$$

which is convergent for $\|u\|_{H^1} < r_m/C_m$. The solution $u := u_\infty$ satisfies

$$-(p(x)u_x)_x = \mu f(x,t,u) - \rho(x)u_{tt} \qquad (36)$$

and $u(\cdot\,,t) \in H^3(0,\pi)$, $\forall t$. Suppose $\|u\|_{H^1} < r_m/C_m$. By induction, assume $u(\cdot\,,t) \in H^k$ for $k = 3,\ldots,m$. Hence $u_{tt}(\cdot\,,t) \in H^k$ and by (35) $f(x,t,u) \in H^k$. This implies by (36) that $u \in H^{k+2}$. ■

Remark 3. *The solution u_∞ is small if $v(\mu,0) = 0$, because $\|u_\infty\|_{H^1} \leq \|u_\infty\|_{\sigma_0/2} \leq K\mu/\gamma\omega$. In this case $u_\infty(\cdot\,,t) \in H^{m+2}$ for $\mu/\gamma\omega$ small enough.*

3.2 Whitney $\mathcal{C}^\infty$ extension

The functions h_n constructed in Lemmas 6 and 7 depend smoothly on the parameters (μ,ω).

Lemma 9. *There is K_4, K_4' such that for $\mu/\gamma^3\omega < K_4'$, the map $h_i(\mu,\omega) \in \mathcal{C}^\infty(A_i, W^{(i)})$, and*

$$\|\partial_\omega h_i(\mu,\omega)\|_{\sigma_i} \leq \frac{K_4\mu}{\gamma^2\omega}\exp(-\chi_0^i), \quad \|\partial_\mu h_i(\mu,\omega)\|_{\sigma_i} \leq \frac{K_4}{\gamma\omega}\exp(-\chi_0^i)$$

where $\chi_0 := (1+\chi)/2$.

Proof. Since $w_0 = \mu L_\omega^{-1} P_0 \mathcal{F}(\mu, w_0)$, by the implicit function theorem the map $w_0(\mu,\omega) \in \mathcal{C}^\infty(A_0, W^{(0)})$. Differentiating the identity $L_\omega(L_\omega^{-1}h) = h$ w.r.t. ω, by (27), we get $\|\partial_\omega L_\omega^{-1}h\|_{\sigma_0} \leq (K/\gamma^2\omega)\,\|h\|_{\sigma_0}$. For $\mu/\gamma\omega$ small,

$$\|\partial_\omega w_0\|_{\sigma_0} \leq \frac{K\mu}{\gamma^2\omega}\,.$$

Differentiating w.r.t. μ we get also $\|\partial_\mu w_0\|_{\sigma_0} \leq K'/\gamma\omega$.

By induction, suppose that h_i depends smoothly on $(\mu,\omega) \in A_i$ for every $i = 0,\ldots,n$. For $(\mu,\omega) \in A_{n+1}$, by (31), h_{n+1} is a solution of

$$-L_\omega h_{n+1} + \mu P_{n+1}[\mathcal{F}(w_n + h_{n+1}) - \mathcal{F}(w_n)] + \mu(P_{n+1} - P_n)\mathcal{F}(w_n) = 0\,. \quad (37)$$

By the implicitic function theorem $h_{n+1} \in \mathcal{C}^\infty$ once we prove that

$$\mathcal{L}_{n+1}(w_{n+1})[z] := -L_\omega z + \mu P_{n+1} D\mathcal{F}(w_n + h_{n+1})[z]$$

is invertible. By (33), $\mathcal{L}_{n+1}(w_n)$ is invertible. Hence it is sufficient that

$$\left\|\mathcal{L}_{n+1}^{-1}(w_n)(\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n))\right\|_{\sigma_{n+1}} < \frac{1}{2}\,,$$

which holds true for $\mu^2/\gamma\omega$ small enough, since, by (30),

$$\|\mathcal{L}_{n+1}(w_{n+1}) - \mathcal{L}_{n+1}(w_n)\|_{\sigma_{n+1}} \leq K\mu\|h_{n+1}\|_{\sigma_{n+1}} \leq \frac{\mu^2 K' N_0^{\tau-1}}{\gamma\omega}\exp(-\chi^{n+1})\,.$$

Finally (33) implies

$$\|\mathcal{L}_{n+1}(w_{n+1})^{-1}\|_{\sigma_{n+1}} \le \frac{2K_1 N_{n+1}^{\tau-1}}{\gamma\omega}\,. \tag{38}$$

Differentiating (37) w.r.t. ω

$$\begin{aligned}\mathcal{L}_{n+1}(w_{n+1})[\partial_\omega h_{n+1}] = \ & 2\omega\rho(x)(h_{n+1})_{tt} - \mu(P_{n+1}-P_n)D\mathcal{F}(w_n)\partial_\omega w_n \\ & - \mu P_{n+1}\big[D\mathcal{F}(w_n+h_{n+1}) - D\mathcal{F}(w_n)\big]\partial_\omega w_n \end{aligned} \tag{39}$$

and, using (38) and (20),

$$\begin{aligned}\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \ \le \ & \frac{KN_{n+1}^{\tau-1}}{\gamma\omega}\Big(\omega N_{n+1}^2\|h_{n+1}\|_{\sigma_{n+1}} + \frac{\mu\|\partial_\omega w_n\|_{\sigma_n}}{\exp[(\sigma_n-\sigma_{n+1})N_n]} + \\ & + \mu\|h_{n+1}\|_{\sigma_{n+1}}\,\|\partial_\omega w_n\|_{\sigma_n}\Big)\,.\end{aligned}$$

We note that $\|\partial_\omega w_n\|_{\sigma_n} \le \sum_{i=0}^n \|\partial_\omega h_i\|_{\sigma_i}$. Using (34) the sequence $a_n := \|\partial_\omega h_n\|_{\sigma_n}$ satisfies

$$\begin{aligned} a_{n+1} \ & \le \ \frac{KN_{n+1}^{\tau-1}}{\gamma\omega}\Big(\omega N_{n+1}^2 r_{n+1} + \frac{\omega\gamma r_{n+1}}{N_{n+1}^{\tau-1}}\sum_{i=0}^n a_i + \mu r_{n+1}\sum_{i=0}^n a_i\Big) \\ & \le \ b_{n+1}\Big(1+\sum_{i=0}^n a_i\Big) \quad \text{where} \quad b_{n+1} := \frac{K\mu}{\gamma^2\omega}\,N_{n+1}^{\tau+1}\,\exp(-\chi^{n+1}),\end{aligned}$$

recalling that $r_{n+1} = (\mu K/\gamma\omega)\exp(-\chi^{n+1})$. By induction, for $K\mu/\omega\gamma^2 < 1$, we have $a_n \le 2b_n$ and

$$\|\partial_\omega h_{n+1}\|_{\sigma_{n+1}} \le \frac{K\mu}{\gamma^2\omega}\,N_{n+1}^{\tau+1}\exp(-\chi^{n+1}) \le \frac{K'\mu}{\gamma^2\omega}\,\exp(-\chi_0^{n+1})$$

where $\chi_0 := (1+\chi)/2$. It follows that $\|\partial_\omega w_{n+1}\|_{\sigma_{n+1}} \le K\mu/\gamma^2\omega$.

Differentiating (37) w.r.t. μ we obtain the estimate for $\partial_\mu h_{n+1}$. ■

Define, for $\nu_0 > 0$,

$$A_n^* := \Big\{(\mu,\omega)\in A_n \ : \ dist\big((\mu,\omega),\,\partial A_n\big) > \frac{\nu_0\gamma^4}{N_n^3}\Big\}$$

$$\tilde{A}_n := \Big\{(\mu,\omega)\in A_n \ : \ dist\big((\mu,\omega),\,\partial A_n\big) > \frac{2\nu_0\gamma^4}{N_n^3}\Big\} \subset A_n^*\,.$$

Lemma 10. ***(Whitney extension)*** *There exists* $\tilde{w} \in \mathcal{C}^\infty(A_0, W \cap X_{\sigma_0/2})$ *satisfying*

$$\|\tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{K_3 \mu}{\gamma\omega}\,, \quad \|\partial_\omega \tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{C\mu}{\gamma^5\omega}\,, \quad \|\partial_\mu \tilde{w}(\mu,\omega)\|_{\sigma_0/2} \le \frac{C}{\gamma^5\omega} \tag{40}$$

for some $C := C(\nu_0) > 0$*, such that,* $\forall (\mu,\omega) \in \tilde{A}_\infty := \bigcap_{n\ge 0} \tilde{A}_n$*,* $\tilde{w}(\mu,\omega)$ *solves the range equation* (19).

Moreover there exists a sequence $\tilde{w}_n \in \mathcal{C}^\infty(A_0, W^{(n)})$ *such that* $\tilde{w}_n(\mu,\omega) = w_n(\mu,\omega)$*,* $\forall (\mu,\omega) \in \tilde{A}_n$*, and*

$$\|\tilde{w}(\mu,\omega) - \tilde{w}_n(\mu,\omega)\|_{\sigma_0/2} \le \frac{K_5 \mu}{\gamma\omega} \exp(-\chi^n)\,. \tag{41}$$

Proof. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ be a $\mathcal{C}^\infty$ function supported in the open ball $B(0,1)$ of center 0 and radius 1 and with $\int_{\mathbb{R}^2} \varphi = 1$. Let $\varphi_n : \mathbb{R}^2 \to \mathbb{R}^+$ be the mollifier

$$\varphi_n(x) := \frac{N_n^6}{\nu_0^2 \gamma^8}\, \varphi\Big(\frac{N_n^3}{\nu_0 \gamma^4}\, x \Big)\,.$$

$\mathrm{Supp}\,(\varphi_n) \subset B(0, \nu_0\gamma^4/N_n^3)$ and $\int_{\mathbb{R}^2} \varphi_n = 1$. We define $\psi_n : \mathbb{R}^2 \to \mathbb{R}$ as

$$\psi_n(x) := \big(\varphi_n * \chi_{A_n^*}\big)(x) = \int_{\mathbb{R}^2} \varphi_n(y-x)\chi_{A_n^*}(y)\, dy$$

where $\chi_{A_n^*}$ is the characteristic function of the set A_n^*. ψ_n is $\mathcal{C}^\infty$,

$$|D\psi_n(x)| \le \int_{\mathbb{R}^2} |D\varphi_n(x-y)|\, \chi_{A_n^*}(y)\, dy \le \frac{N_n^3}{\nu_0\gamma^4}\, C \tag{42}$$

where $C := \int_{\mathbb{R}^2} |D\varphi|\, dy$,

$$0 \le \psi_n(x) \le 1, \qquad \mathrm{supp}(\psi_n) \subset A_n, \qquad \psi_n(x) = 1 \ \ \forall\, x \in \tilde{A}_n\,.$$

We define, for $(\mu,\omega) \in A_0$, the C^∞ functions

$$\tilde{h}_n(\mu,\omega) := \begin{cases} \psi_n(\mu,\omega) h_n(\mu,\omega) & \text{if} \quad (\mu,\omega) \in A_n \\ 0 & \text{if} \quad (\mu,\omega) \notin A_n \end{cases}$$

and

$$\tilde{w}_n(\mu,\omega) := \sum_{i=0}^n \tilde{h}_i\,, \qquad \tilde{w}(\mu,\omega) := \sum_{i\ge 0} \tilde{h}_i$$

which is a series if $(\mu,\omega) \in A_\infty := \bigcap_{n\ge 0} A_n$.

The estimate for $\|\tilde{w}\|_{\sigma_0/2}$ follows by $\|\tilde{h}_i\|_{\sigma_i} \leq \|h_i\|_{\sigma_i}$ (because $0 \leq \psi_i \leq 1$) and (28). The estimates for the derivatives in (40) follow differentiating the product $\tilde{h}_i = \psi_i h_i$ and using (42), (28) and Lemma 9. Similarly it follows that $\tilde{w}$ is in C^∞, see [7] for details.

For $(\mu, \omega) \in \tilde{A}_n$, $\psi_n(\mu, \omega) = 1$, implying $\tilde{w}_n = w_n$. As a consequence, for $(\mu, \omega) \in \tilde{A}_\infty := \bigcap_{n \geq 0} \tilde{A}_n$, by Corollary 1, $\tilde{w} = w_\infty$ solves (19).

Finally, using (28),

$$\|\tilde{w} - \tilde{w}_n\|_{\sigma_0/2} \leq \sum_{i \geq n+1} \|\tilde{h}_i\|_{\sigma_i} \leq \sum_{i \geq n+1} \frac{K\mu}{\gamma\omega} \exp(-\chi^i) \leq \frac{K'\mu}{\gamma\omega} \exp(-\chi^n)\,.$$

■

Lemma 11. *There exist K_5' such that if $\mu/\gamma^2\omega < K_5'$ and $\nu_0 < K_5'$ then*

$$B_\gamma \subseteq \tilde{A}_n \quad \forall\, n \geq 0$$

where B_γ is defined in (11)*.*

Proof. By induction. Let $(\mu, \omega) \in B_\gamma$. Then $(\mu, \omega) \in \tilde{A}_0$ if A_0 contains the closed ball of center (μ, ω) and radius $2\nu_0\gamma^4/N_0^3$. Let (ω', μ') belong to such a ball. Then, $\forall l = 1, \ldots, N_0$,

$$|\omega' l - \omega_j| \geq |\omega l - \omega_j| - |\omega - \omega'| l > \frac{2\gamma}{l^\tau} - \frac{2\nu_0\gamma^4}{N_0^3}\, l \geq \frac{\gamma}{l^\tau}$$

if $\nu_0 \leq 1/2$.

Suppose now $B_\gamma \subseteq \tilde{A}_n$ and let $(\mu, \omega) \in B_\gamma$. To prove that $(\mu, \omega) \in \tilde{A}_{n+1}$, we have to show that the closed ball of center (μ, ω) and radius $2\nu_0\gamma^4/N_{n+1}^3$ is contained in A_{n+1}. Let (μ', ω') belong to such a ball. The non-resonance condition on $|\omega' l - j/c|$ is verified, as above, for $\nu_0 \leq 1/2$. For the other condition, we denote in short $\omega_j^n(\mu', \omega') := \omega_j(\mu', w_n(\mu', \omega'))$ (see (16) for the definition of $\omega_j(\mu, w)$). It results, $\forall l = 1, \ldots, N_{n+1}$,

$$\begin{aligned}
|\omega' l - \omega_j^n(\mu', \omega')| \;&\geq\; |\omega l - \tilde{\omega}_j(\mu, \omega)| - |\omega - \omega'| l - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \\
&>\; \frac{2\gamma}{l^\tau} - \frac{2\nu_0\gamma^4 l}{N_{n+1}^3} - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \\
&>\; \frac{3\gamma}{2l^\tau} - |\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| \qquad (43)
\end{aligned}$$

if $\nu_0 \leq 1/4$. We now estimate the last term

$$|\omega_j^n(\mu', \omega') - \tilde{\omega}_j(\mu, \omega)| = \frac{|\lambda_j^n(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{|\tilde{\omega}_j(\mu, \omega)| + |\omega_j^n(\mu', \omega')|} \leq \frac{|\lambda_j^n(\mu', \omega') - \tilde{\lambda}_j(\mu, \omega)|}{\sqrt{\delta_0}}$$

by (18), both for $j < j_0$ and for $j \geq j_0$. By the comparison principle (17)

$$\delta_0^{-1/2}|\lambda_j^n(\mu',\omega') - \tilde{\lambda}_j(\mu,\omega)| \leq K|\mu - \mu'| + K\|w_n(\mu',\omega') - \tilde{w}(\mu,\omega)\|_{\sigma_0/2}\,.$$

By Lemma 9, $\|\partial_\omega w_n\|_{\sigma_0/2}$, $\|\partial_\mu w_n\|_{\sigma_0/2} \leq K_0/\omega\gamma^2$, and being $\omega, \omega' > \gamma$,

$$K\|w_n(\mu',\omega') - w_n(\mu,\omega)\|_{\sigma_0/2} \leq \frac{K'}{\gamma^3}\,\frac{\nu_0\gamma^4}{N_{n+1}^3} < \frac{\gamma}{8l^\tau}\,,\ \forall l = 1,\ldots,N_{n+1}$$

if ν_0 is small enough ($1 < \tau < 2$). On the other hand, since $(\mu,\omega) \in \tilde{A}_n$ we have $w_n(\mu,\omega) = \tilde{w}_n(\mu,\omega)$ (Lemma 10) and, by (41),

$$K\|w_n(\mu,\omega) - \tilde{w}(\mu,\omega)\|_{\sigma_0/2} \leq \frac{K''\mu}{\gamma\omega}\exp(-\chi^n) < \frac{\gamma}{8l^\tau}\,,\ \forall l = 1,\ldots,N_{n+1}$$

for $\mu/\gamma^2\omega$ sufficiently small. By (43), collecting the previous estimates,

$$|\omega' l - \omega_j^n(\mu',\omega')| > \frac{\gamma}{l^\tau}\,, \quad \forall l = 1,\ldots,N_{n+1}$$

and (μ',ω') belongs to A_{n+1}. ■

3.3 Measure of the Cantor set B_γ

In the following $R := (\mu',\mu'') \times (\omega',\omega'')$ denotes a rectangle contained in the region $\{(\mu,\omega) \in [\mu_1,\mu_2] \times (2\gamma,+\infty) : \mu < K_6'\gamma^5\omega\}$. Furthermore we consider $\omega'' - \omega'$ as a fixed quantity ("of order 1").

Lemma 12. *There exist K_6, K_6' such that, $\forall \mu \in [\mu_1,\mu_2]$, the section*

$$S_\gamma(\mu) := \{\omega : (\mu,\omega) \in B_\gamma\}$$

with $\mu/\omega\gamma^5 < K_6'$ in the definition (11) *of B_γ, satisfies the measure estimate*

$$|S_\gamma(\mu) \cap (\omega',\omega'')| \ \geq (1 - K_6\gamma)(\omega'' - \omega') \tag{44}$$

for γ small. As a consequence, for every $R := (\mu',\mu'') \times (\omega',\omega'')$

$$|B_\gamma \cap R| \ \geq \ |R|\,(1 - K_6\gamma)\,. \tag{45}$$

Proof. We consider just the inequalities $|\omega l - \tilde{\omega}_j(\mu,\omega)| > 2\gamma/l^\tau$ in the definition of B_γ. The analogous inequalities are simpler because j/c and ω_j do not depend on (μ,ω).

The complementary set we have to estimate is

$$\mathcal{C} := \bigcup_{l,j\geq 1} \mathcal{R}_{lj}$$

where $\mathcal{R}_{lj} := \{\omega \in (\omega', \omega'') \,:\, |l\omega - \tilde{\omega}_j(\mu,\omega)| \leq 2\gamma/l^\tau\}$.

We claim that

$$\big|\partial_\omega \tilde{\omega}_j(\mu,\omega)\big| \leq \frac{K\mu}{\gamma^5\omega}\,. \tag{46}$$

Indeed, by the same arguments as in the proof of Lemma 11 and the comparison principle (17) we have

$$|\tilde{\omega}_j(\mu,\omega) - \tilde{\omega}_j(\mu,\omega')| \leq K\|\tilde{w}(\mu,\omega) - \tilde{w}(\mu,\omega')\|_{\sigma_0/2} \leq \frac{K\mu}{\gamma^5\omega}\,|\omega - \omega'|$$

using (40). As a consequence of (46)

$$\partial_\omega\big(l\omega - \tilde{\omega}_j(\mu,\omega)\big) \geq l - \frac{K\mu}{\gamma^5\omega} \geq \frac{l}{2} \qquad \forall l \geq 1$$

for $\mu/\gamma^5\omega$ small enough; we deduce $|\mathcal{R}_{lj}| \leq 4\gamma/l^{\tau+1}$.

Furthermore the set $\mathcal{R}_{lj}$ is non-empty only if

$$\omega' l - \frac{2\gamma}{l^\tau} < \tilde{\omega}_j(\mu,\omega) < \omega'' l + \frac{2\gamma}{l^\tau}\,.$$

So, for every fixed l, the number of indices j such that $\mathcal{R}_{l,j} \neq \emptyset$ is

$$\sharp\{j\} \leq \frac{1}{\delta}\left(l(\omega''-\omega') + \frac{4\gamma}{l^\tau}\right) + 1 \leq Kl(\omega''-\omega')$$

where

$$\delta := \inf\left\{\, |\tilde{\omega}_{j+1}(\mu,\omega) - \tilde{\omega}_j(\mu,\omega)| \,:\, j \geq 1,\ (\mu,\omega) \in B_\gamma \right\}.$$

For $\|\tilde{w}\|_{\sigma_0/2} \leq K'\mu/\gamma\omega < R$ we have $\delta \geq \delta_1$ where

$$\delta_1 := \inf\left\{\big|\omega_{j+1}(\mu,w) - \omega_j(\mu,w)\big| : \ j \geq 1\,,\ \mu \in [\mu_1,\mu_2],\ \|w\|_{\sigma_0/2} \leq R\right\} > 0 \tag{47}$$

as proved in the Appendix.

In conclusion, the measure of the complementary set is

$$|\mathcal{C}| \leq \sum_{l=1}^{+\infty} \frac{4\gamma}{l^{\tau+1}}\, K\, l(\omega''-\omega') \leq K'(\omega''-\omega')\gamma$$

and (44) is proved. Integrating on (μ',μ'') we obtain (45). ■

By Fubini Theorem also the section $S_\gamma(\omega)$ is large, for ω in a large set.

Lemma 13. *Let*

$$S_\gamma(\omega) := \{\mu : (\mu, \omega) \in B_\gamma\}.$$

For every $R := (\mu', \mu'') \times (\omega', \omega'')$, $\gamma' \in (0, 1)$ *it results*

$$\Big|\Big\{\omega \in (\omega', \omega'') : \frac{|S_\gamma(\omega) \cap (\mu', \mu'')|}{\mu'' - \mu'} \ge 1 - \gamma'\Big\}\Big| \ge (\omega'' - \omega')\Big(1 - K_6 \frac{\gamma}{\gamma'}\Big). \quad (48)$$

Proof. Let consider

$$\Omega^+ := \big\{\omega \in (\omega', \omega'') : |S_\gamma(\omega) \cap (\mu', \mu'')| \ge (\mu'' - \mu')(1 - \gamma')\big\}$$
$$\Omega^- := \big\{\omega \in (\omega', \omega'') : |S_\gamma(\omega) \cap (\mu', \mu'')| < (\mu'' - \mu')(1 - \gamma')\big\}.$$

Using the Fubini theorem

$$\begin{aligned} |B_\gamma \cap R| &= \int_{\omega'}^{\omega''} |S_\gamma(\omega) \cap (\mu', \mu'')|\, d\omega \qquad (49)\\ &= \int_{\Omega^+} |S_\gamma(\omega) \cap (\mu', \mu'')|\, d\omega + \int_{\Omega^-} |S_\gamma(\omega) \cap (\mu', \mu'')|\, d\omega \\ &\le (\mu'' - \mu')|\Omega^+| + (\mu'' - \mu')(1 - \gamma')|\Omega^-|. \end{aligned}$$

Minorating the left hand side in (49) by (45) yields

$$(\omega'' - \omega')(1 - K_6\gamma) \le |\Omega^+| + (1 - \gamma')|\Omega^-| = (\omega'' - \omega') - \gamma'|\Omega^-| \qquad (50)$$

and therefore $|\Omega^-| \le (\omega'' - \omega')K_6\gamma/\gamma'$. We deduce by the first inequality in (50) that $|\Omega^+| \ge (\omega'' - \omega')(1 - K_6\gamma/\gamma')$, namely (48). ■

By (44)-(48) we deduce the measure estimates for the "sections" (in ω and μ) of $\cup_{\gamma\in(0,1)} B_\gamma$ stated after Theorem 1.

4 Inversion of the linearized problem

Here we prove Lemma 5. Decomposing in Fourier series $f'(u) = \sum_{k\in\mathbb{Z}} a_k(x)$ e^{ikt} we have, $\forall h = \sum_{1\le|l|\le N_n} h_l(x)e^{ilt} \in W^{(n)}$,

$$\begin{aligned} -L_\omega h + \mu P_n[f'(u)h] &= \sum_{1\le|l|\le N_n} \big[\omega^2 l^2 \rho h_l + \partial_x\big(p\,\partial_x h_l\big)\big]e^{ilt} + \\ &\quad + \mu P_n\big[\big(\sum_{k\in\mathbb{Z}} a_k e^{ikt}\big)\big(\sum_{1\le|l|\le N_n} h_l e^{ilt}\big)\big] \\ &= \sum_{1\le|l|\le N_n} \big[\omega^2 l^2 \rho h_l + \partial_x\big(p\,\partial_x h_l\big) + \mu a_0 h_l\big]e^{ilt} \qquad (51)\\ &\quad + \mu \sum_{|l|,|k+l|\in\{1,\dots,N_n\},\, k\ne 0} a_k\, h_l\, e^{i(k+l)t} \qquad (52) \end{aligned}$$

distinguishing the diagonal operator (51) by the off-diagonal term (52). Hence $\mathcal{L}_n(w)$ defined in (21) can be decomposed as

$$\mathcal{L}_n(w)h = \rho\big(Dh + M_1 h + M_2 h\big) \tag{53}$$

where

$$\begin{aligned} Dh &:= \frac{1}{\rho}\sum_{|l|=1}^{N_n} \big[\omega^2 l^2 \rho\, h_l + \big(p\, h_l'\big)' + \mu a_0\, h_l\big] e^{ilt} \\ M_1 h &:= \frac{\mu}{\rho} \sum_{|l|,|k|\in\{1,\ldots,N_n\}, l\neq k} a_{k-l}\, h_l\, e^{ikt} \\ M_2 h &:= \frac{\mu}{\rho}\, P_n\big[f'(u)\, d_w v(\mu,w)[h]\big]\,. \end{aligned} \tag{54}$$

To study the eigenvalues of D, we use Sturm-Liouville type techniques.

Lemma 14. ***(Sturm-Liouville)*** *The eigenvalues $\lambda_j(\mu,w)$ of the Sturm-Liouville problem (16) form a strictly increasing sequence which tends to $+\infty$. Every $\lambda_j(\mu,w)$ is simple and the following asymptotic formula holds*

$$\lambda_j(\mu,w) = \frac{j^2}{c^2} + b + M(\mu,w) + r_j(\mu,w), \qquad |r_j(\mu,w)| \le \frac{K}{j} \tag{55}$$

$\forall j \ge 1$, $(\mu,w) \in [\mu_1,\mu_2]\times B_R$, where

$$\begin{aligned} c &:= \frac{1}{\pi}\int_0^\pi \Big(\frac{\rho}{p}\Big)^{1/2} dx, \qquad b := \frac{1}{4\pi c}\int_0^\pi \bigg[\frac{(\rho p)'}{\rho\,\sqrt[4]{\rho p}}\bigg]' \frac{1}{\sqrt[4]{\rho p}}\, dx, \\ M(\mu,w) &:= -\frac{\mu}{c\pi}\int_0^\pi \frac{\Pi_V f'(v(\mu,w)+w)}{\sqrt{\rho p}}\, dx\,. \end{aligned}$$

The eigenfunctions $\varphi_j(\mu,w)$ of (16) form an orthonormal basis of $L^2(0,\pi)$ with respect to the scalar product $(y,z)_{L^2_\rho} := c^{-1}\int_0^\pi yz\rho\, dx$. For K big enough

$$(y,z)_{\mu,w} := \frac{1}{c}\int_0^\pi p\, y'z' + \big[K\rho - \mu\Pi_V f'(v(\mu,w)+w)\big]\, yz\, dx$$

defines an equivalent scalar product on $H_0^1(0,\pi)$ and

$$K'\|y\|_{H^1} \le \|y\|_{\mu,w} \le K''\,\|y\|_{H^1} \quad \forall y \in H_0^1\,. \tag{56}$$

$\varphi_j(\mu,w)$ is also an orthogonal basis of $H_0^1(0,\pi)$ with respect to the scalar product $(\ ,\ )_{\mu,w}$ and, for $y = \sum_{j\ge1}\hat y_j\,\varphi_j(\mu,w)$,

$$\|y\|^2_{L^2_\rho} = \sum_{j\ge1}\hat y_j^2\,, \qquad \|y\|^2_{\mu,w} = \sum_{j\ge1}\hat y_j^2\big(\lambda_j(\mu,w)+K\big)\,. \tag{57}$$

Proof. In the Appendix. ■

We develop $Dh = \sum_{1\le|l|\le N_n} D_l h_l \, e^{ilt}$ where

$$D_l z := \frac{1}{\rho}\left[\omega^2 l^2 \rho z + \left(p\, z'\right)' + \mu a_0 z\right], \quad \forall\, z \in H_0^1(0,\pi)$$

and $a_0 = \Pi_V f(v(\mu,w)+w)$.

By Lemma 14 each D_l is diagonal w.r.t the basis $\varphi_j(\mu,w)$:

$$z = \sum_{j=1}^{+\infty} \hat{z}_j \, \varphi_j(\mu,w) \in H_0^1(0,\pi) \quad \Rightarrow \quad D_l z = \sum_{j=1}^{+\infty} \left(\omega^2 l^2 - \lambda_j(\mu,w)\right) \hat{z}_j \, \varphi_j(\mu,w)\,.$$

Lemma 15. *Suppose all the eigenvalues $\omega^2 l^2 - \lambda_j(\mu,w)$ are not zero. Then*

$$|D_l|^{-1/2} z := \sum_{j=1}^{+\infty} \frac{\hat{z}_j \, \varphi_j(\mu,w)}{\sqrt{|\omega^2 l^2 - \lambda_j(\mu,w)|}}$$

satisfies

$$\big\| \, |D_l|^{-1/2} z \big\|_{H^1} \le \frac{K}{\sqrt{\alpha_l}} \, \|z\|_{H^1}\,, \quad \forall z \in H_0^1(0,\pi) \tag{58}$$

where $\alpha_l := \min_{j\ge1} |\omega^2 l^2 - \lambda_j(\mu,w)| > 0$.

Proof. By (57) $\| \, |D_l|^{-1/2} z\|^2_{\mu,w} \le (1/\alpha_l)\, \|z\|^2_{\mu,w}$. Hence (58) follows by the equivalence of the norms (56). ■

Lemma 16. ***(Inversion of** D**)*** *Assume the non-resonance condition (23). Then $|D|^{-1/2} : W^{(n)} \to W^{(n)}$ defined by*

$$|D|^{-1/2} h := \sum_{1\le|l|\le N_n} |D_l|^{-1/2} h_l \, e^{ilt}$$

satisfies

$$\| \, |D|^{-1/2} h\|_{\sigma,s} \le \frac{K}{\sqrt{\gamma\omega}} \, \|h\|_{\sigma,s+\frac{\tau-1}{2}} \le \frac{K N_n^{\frac{\tau-1}{2}}}{\sqrt{\gamma\omega}} \, \|h\|_{\sigma,s}\,, \quad \forall\, h \in W^{(n)}\,.$$

Proof. By (58) and $\alpha_{-l} = \alpha_l \ge \gamma\omega/|l|^{\tau-1}$

$$\begin{aligned} \| \, |D|^{-1/2} h\|^2_{\sigma,s} &= \sum_{1\le|l|\le N_n} \| \, |D_l|^{-1/2} h_l\|^2_{H^1} (1+l^{2s}) e^{2\sigma|l|} \\ &\le \sum_{1\le|l|\le N_n} \frac{K^2 |l|^{\tau-1}}{\gamma\omega} \, \|h_l\|^2_{H^1} (1+l^{2s}) e^{2\sigma|l|} \\ &\le \frac{K'}{\gamma\omega} \|h\|^2_{\sigma,s+\frac{\tau-1}{2}} \end{aligned}$$

because $|l|^{\tau-1}(1+l^{2s}) < 2(1+|l|^{2s+\tau-1})$, $\forall\, |l| \geq 1$. ■

To prove the invertibility of $\mathcal{L}_n(w)$ we write (53) as

$$\mathcal{L}_n(w) = \rho |D|^{1/2}\big(U + T_1 + T_2\big)|D|^{1/2} \tag{59}$$

where

$$\begin{cases} U := |D|^{-1/2}\, D\, |D|^{-1/2} \\ T_i := |D|^{-1/2}\, M_i\, |D|^{-1/2}\,, \quad i = 1, 2\,. \end{cases} \tag{60}$$

With respect to the basis $\varphi_j(\mu, w)\, e^{ilt}$ the operator U is diagonal and its (l, j)-th eigenvalue is $\mathrm{sign}(\omega^2 l^2 - \lambda_j(\mu, w)) \in \{\pm 1\}$, implying[5] $\|U\|_\sigma = 1$.

The smallness of T_1 requires an analysis of the small divisors. Formula (55) implies, by Taylor expansion, the asymptotic dispersion relation

$$\Big|\omega_j(\mu, w) - \frac{j}{c}\Big| \leq \frac{K}{j} \tag{61}$$

and there exists K such that, for every $x \geq 0$,

$$|x^2 - \lambda_{j^*}(\mu, w)| = \min_{j \geq 1} |x^2 - \lambda_j(\mu, w)| \quad \Rightarrow \quad j^* \geq Kx\,. \tag{62}$$

Lemma 17. *Assume the non-resonance conditions (22)-(23) and $\omega > \gamma$. Then $\forall |k|, |l| \in \{1, \ldots, N_n\}$, $k \neq l$*

$$\alpha_l \alpha_k \geq \left(\frac{K\gamma^3\omega}{|k-l|^{\frac{\tau(\tau-1)}{2-\tau}}}\right)^2$$

where $\alpha_l := \min_{j \geq 1} |\omega^2 l^2 - \lambda_j(\mu, w)|$.

Proof. Since $\alpha_{-l} = \alpha_l$, $\forall l$, we can suppose $l, k \geq 1$.

We distinguish two cases, if k, l are close or far one from each other. Let $\beta := (2-\tau)/\tau \in (0, 1)$.

Case 1. Let $2|k-l| > (\max\{k, l\})^\beta$. By (23)

$$\alpha_k \alpha_l \geq \frac{(\gamma\omega)^2}{(kl)^{\tau-1}} \geq \frac{(\gamma\omega)^2}{(\max\{k, l\})^{2(\tau-1)}} \geq \frac{C(\gamma\omega)^2}{|k-l|^{\frac{2(\tau-1)}{\beta}}}\,.$$

Case 2. Let $0 < 2|k-l| \leq (\max\{k, l\})^\beta$. In this case $2k \geq l \geq k/2$. Indeed, if

[5] The operator norm is $\|U\|_\sigma := \sup_{\|h\|_\sigma \leq 1} \|Uh\|_\sigma$.

$k > l$, then $2(k-l) \leq k^\beta$, so $2l \geq 2k - k^\beta \geq k$ because $\beta \in (0,1)$. Analogously if $l > k$.

Let i, resp. j, be an integer which realizes the minimum α_k, resp. α_l, and write in short $\lambda_j(\mu) := \lambda_j(\mu, w)$, $\omega_j(\mu) := \omega_j(\mu, w)$.

If both $\lambda_i(\mu), \lambda_j(\mu) \leq 0$, then $\alpha_l \geq \omega^2 l^2$, $\alpha_k \geq \omega^2 k^2$, $\alpha_l \alpha_k \geq \omega^4 > \gamma^2\omega^2$.

If only $\lambda_j(\mu) \leq 0$, then $\alpha_l \alpha_k \geq \gamma\omega^3 l^2 / k^{\tau-1} \geq 2^{1-\tau}\gamma\omega^3 \geq 2^{1-\tau}\gamma^2\omega^2$.

The really resonant cases happen if $\lambda_i(\mu), \lambda_j(\mu) > 0$.

Suppose, for example, $\max\{k,l\} = k$. By (61), $|\omega_j(\mu) - (j/c)| \leq K/j$, and, by (62), $i \geq K\omega k$, $j \geq K\omega l$. Hence, using also (22),

$$
\begin{aligned}
\left|(\omega k - \omega_i(\mu)) - (\omega l - \omega_j(\mu))\right| &= \Big|\omega(k-l) - (\omega_i(\mu) - \omega_j(\mu))\Big| \\
&\geq \Big|\omega(k-l) - \frac{i-j}{c}\Big| - \frac{K}{\omega l} - \frac{K}{\omega k} \\
&\geq \frac{\gamma}{(k-l)^\tau} - \frac{3K}{\omega k} \geq \frac{2^\tau \gamma}{k^{\beta\tau}} - \frac{3K}{\omega k}
\end{aligned}
$$

because $2(k-l) \leq k^\beta$, $2l \geq k$. Since $\beta\tau < 1$ and $k \leq 2l$,

$$
\Big|(\omega k - \omega_i(\mu)) - (\omega l - \omega_j(\mu))\Big| \geq \frac{1}{2}\Big(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\Big) \qquad \forall\, k \geq \Big(\frac{K}{\omega\gamma}\Big)^{\frac{1}{1-\beta\tau}} := k^* \,.
$$

The same conclusion if $\max\{k,l\} = l$. It follows that, for $\max\{k,l\} \geq k^*$, there holds $|\omega k - \omega_i(\mu)| \geq \gamma/2k^{\beta\tau}$ or $|\omega l - \omega_j(\mu)| \geq \gamma/2l^{\beta\tau}$. Suppose $|\omega k - \omega_i(\mu)| \geq \gamma/2k^{\beta\tau}$. Then

$$
\alpha_k = |\omega^2 k^2 - \omega_i^2(\mu)| \geq |\omega k - \omega_i(\mu)|\omega k \geq \frac{\gamma\omega}{2}\, k^{1-\beta\tau} \,.
$$

Since $l \leq 2k$, for α_l we can use (23),

$$
\alpha_k \alpha_l \geq \frac{\gamma\omega k^{1-\beta\tau}}{2}\,\frac{\gamma\omega}{l^{\tau-1}} \geq \frac{\gamma^2\omega^2}{2^\tau}\, k^{2-\tau-\beta\tau} = \frac{\gamma^2\omega^2}{2^\tau}
$$

because $2 - \tau - \beta\tau = 0$.

On the other hand, if $\max\{k,l\} < k^* = (K/\omega\gamma)^{1/(\tau-1)}$, we can use (23) for both k, l:

$$
\alpha_k \alpha_l \geq \frac{(\gamma\omega)^2}{(kl)^{\tau-1}} > \frac{(\gamma\omega)^2}{(k^*)^{2(\tau-1)}} = (\gamma\omega)^2 \left(\frac{\omega\gamma}{K}\right)^{\frac{1}{\tau-1}\,2(\tau-1)} > \frac{\gamma^6\omega^2}{K^2} \,.
$$

Since $\gamma < 1$, taking the minimum for all these cases we conclude. ■

Lemma 18. ***(Estimate of*** T_1***)*** *Assume the non-resonance conditions (22)-(23), $\omega > \gamma$, and $\Pi_W f'(u) = \sum_{l\neq 0} a_l(x) e^{ilt} \in X_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}}$. There exists K such that*

$$\|T_1 h\|_\sigma \leq \frac{K\mu}{\gamma^3\omega} \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}} \|h\|_\sigma, \quad \forall h \in W^{(n)}.$$

Proof. $\forall h \in W^{(n)}$, $T_1 h = \sum_{1\leq |k| \leq N_n} (T_1 h)_k \, e^{ikt}$ where

$$\begin{aligned}
(T_1 h)_k &= |D_k|^{-1/2} \big(M_1 |D|^{-1/2} h \big)_k \\
&= |D_k|^{-1/2} \Big[\sum_{1\leq |l| \leq N_n, \, l\neq k} \mu \, \frac{a_{k-l}}{\rho} \, |D_l|^{-1/2} h_l \Big].
\end{aligned}$$

Setting $A_m := \|a_m/\rho\|_{H^1}$, using (58) and Lemma 17,

$$\|(T_1 h)_k\|_{H^1} \leq K\,\mu \sum_{1\leq |l| \leq N_n, \, l\neq k} \frac{A_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}} \|h_l\|_{H^1} \leq \frac{K\mu}{\gamma^3\omega} S_k \tag{63}$$

where

$$S_k := \sum_{|l|\leq N_n, \, l\neq k} A_{k-l} \, |k-l|^{\frac{\tau(\tau-1)}{2-\tau}} \|h_l\|_{H^1}.$$

By (63) we get, defining $S(t) := \sum_{|k|=1}^{N_n} S_k \, e^{ikt}$,

$$\begin{aligned}
\|T_1 h\|_\sigma^2 &= \sum_{|k|=1}^{N_n} \|(T_1 h)_k\|_{H^1}^2 (1+k^2) e^{2\sigma|k|} \\
&\leq \Big(\frac{K\mu}{\gamma^3\omega}\Big)^2 \sum_{|k|=1}^{N_n} S_k^2 (1+k^2) e^{2\sigma|k|} = \Big(\frac{K\mu}{\gamma^3\omega}\Big)^2 \|S\|_\sigma^2.
\end{aligned}$$

Since $S = P_n(\varphi\psi)$ with $\varphi(t) := \sum_{l\in\mathbb{Z}} A_l |l|^{\frac{\tau(\tau-1)}{2-\tau}} \, e^{ilt}$ and $\psi(t) := \sum_{|l|=1}^{N_n} \|h_l\|_{H^1} \, e^{ilt}$

$$\|T_1 h\|_\sigma \leq \frac{K\mu}{\gamma^3\omega} \|\varphi\|_\sigma \|\psi\|_\sigma \leq \frac{K\mu}{\gamma^3\omega} \Big\|\Pi_W f'(u)\Big\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}} \|h\|_\sigma$$

because $\|\varphi\|_\sigma \leq 2\|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau(\tau-1)}{2-\tau}}$ and $\|\psi\|_\sigma = \|h\|_\sigma$. ■

Lemma 19. ***(Estimate of*** T_2***)*** *Suppose that $\Pi_W f'(u) \in X_{\sigma, 1+\frac{\tau-1}{2}}$. Then*

$$\|T_2 h\|_\sigma \leq \frac{K\mu}{\gamma\omega} \|\Pi_W f'(u)\|_{\sigma, 1+\frac{\tau-1}{2}} \|h\|_\sigma, \qquad \forall\, h \in W^{(n)}$$

for some K.

Proof. By the definitions (60),(54) and Lemma 16,

$$\begin{aligned}
\|T_2 h\|_\sigma &\leq \frac{K}{\sqrt{\gamma\omega}} \|M_2 |D|^{-1/2} h\|_{\sigma, 1+\frac{\tau-1}{2}} \\
&\leq \frac{K'\mu}{\sqrt{\gamma\omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \left\| d_w v(\mu, w)\big[\, |D|^{-1/2} h\big]\, \right\|_{\sigma, 1+\frac{\tau-1}{2}} \\
&= \frac{K'\mu}{\sqrt{\gamma\omega}} \|\Pi_W f'(u)\|_{\sigma,1+\frac{\tau-1}{2}} \left\| d_w v(\mu, w)\big[\, |D|^{-1/2} h\big]\, \right\|_{H^1}
\end{aligned}$$

because $d_w v(\mu, w)[|D|^{-1/2} h] \in V$. By Lemmas 4 and 16

$$\left\| d_w v(\mu, w)\big[\, |D|^{-1/2} h\big]\, \right\|_{H^1} \leq K \|\, |D|^{-1/2} h\|_{\sigma, 1-\frac{\tau-1}{2}} \leq \frac{K}{\sqrt{\gamma\omega}} \|h\|_{\sigma,1}$$

implying the thesis. ■

Proof of Lemma 5. $\|U\|_\sigma = 1$. If $\|T_1 + T_2\|_\sigma < 1/2$, then by Neumann series $U + T_1 + T_2$ is invertible in $(W^{(n)}, \|\ \|_\sigma)$ and $\|(U + T_1 + T_2)^{-1}\|_\sigma < 2$. By Lemmas 18, 19, this condition is verified if we choose K_1 in (24) small enough. Hence, inverting (59)

$$\mathcal{L}_n(w)^{-1} h = |D|^{-1/2} \big(U + T_1 + T_2 \big)^{-1} |D|^{-1/2} \Big(\frac{h}{\rho} \Big)$$

which, using Lemma 16, yields (25). ■

5 Appendix

Proof of Lemma 14. Let $a(x) \in L^2(0, \pi)$. Under the "Liouville change of variable"

$$x = \psi(\xi) \ \Leftrightarrow\ \xi = g(x), \qquad g(x) := \frac{1}{c} \int_0^x \Big(\frac{\rho(s)}{p(s)} \Big)^{1/2} ds\,, \tag{64}$$

we have that $(\lambda, y(x))$ satisfies

$$\begin{cases} -(p(x) y'(x))' + a(x) y(x) = \lambda \rho(x) y(x) \\ y(0) = y(\pi) = 0 \end{cases} \tag{65}$$

if and only if $(\nu, z(\xi))$ satisfies

$$\begin{cases} -z''(\xi) + [q(\xi) + \alpha(\xi)]\, z(\xi) = \nu z(\xi) \\ z(0) = z(\pi) = 0 \end{cases} \tag{66}$$

where

$$\nu = c^2\lambda, \qquad r(x) = \sqrt[4]{p(x)\,\rho(x)}, \qquad z(\xi) = y(\psi(\xi))\,r(\psi(\xi)),$$
$$\alpha(\xi) = c^2\,\frac{a(\psi(\xi))}{\rho(\psi(\xi))}\,, \qquad q(\xi) = c^2 Q(\psi(\xi)), \qquad Q = \frac{p}{\rho}\,\frac{r''}{r} + \frac{1}{2}\Big(\frac{p}{\rho}\Big)'\,\frac{r'}{r}\,.$$

By [20], Theorem 4 in Chapter 2, p.35, the eigenvalues of (66) form an increasing sequence ν_j satisfying the asympototics

$$\nu_j = j^2 + \frac{1}{\pi}\int_0^\pi (q+\alpha)\,d\xi \, - \frac{1}{\pi}\int_0^\pi \cos(2j\xi)\big(q(\xi)+\alpha(\xi)\big)\,d\xi + r_j, \quad |r_j| \leq \frac{C}{j}$$

where $C := C(\|q+\alpha\|_{L^2})$ is a positive constant. Moreover every ν_j is simple ([20], Theorem 2, p.30).

Since p, ρ are positive and belong to H^3, if $a \in H^1$ then $q, \alpha \in H^1$. Integrating by parts $|\int_0^\pi \cos(2j\xi)(q+\alpha)\,d\xi| \leq \|q+\alpha\|_{H^1}/j$ and so

$$\nu_j = j^2 + \frac{1}{\pi}\int_0^\pi (q+\alpha)\,d\xi + r'_j, \quad |r'_j| \leq \frac{C'}{j}$$

for some $C' := C'(\|q+\alpha\|_{H^1})$. Dividing by c^2 and using the inverse Liouville change of variable we obtain the formula for the eigenvalues $\lambda_j(a)$ of (65)

$$\lambda_j(a) = \frac{j^2}{c^2} + \frac{1}{\pi c}\int_0^\pi \frac{Q\sqrt{\rho}}{\sqrt{p}}\,dx + \frac{1}{\pi c}\int_0^\pi \frac{a}{\sqrt{\rho p}}\,dx + r_j(a), \quad |r_j(a)| \leq \frac{C}{j} \quad (67)$$

for some $C(\rho, p, \|a\|_{H^1}) > 0$. Formula (55) follows for $a(x) = -\mu\Pi_V f'(v(\mu, w) + w)(x)$ and some algebra.

By [20], Theorem 7 p.43, the eigenfunctions of (66) form an orthonormal basis for L^2. Applying in the integrals the Liouville change of variable, the eigenfunctions $\varphi_j(a)$ of (65) form an orthonormal basis for L^2 w.r.t. the scalar product $(\ ,\ )_{L^2_\rho}$.

Finally, since $\varphi_j := \varphi_j(a)$ solves

$$-(p\varphi'_j)' + (K\rho + a)\varphi_j = (\lambda_j(a) + K)\rho\varphi_j\,,$$

multiplying by φ_i and integrating by parts gives

$$(\varphi_j, \varphi_i)_{\mu,w} = \delta_{i,j}(\lambda_j(a) + K)$$

and (57) follows (note that $\lambda_j(a) + K > 0$, $\forall j$, for K large enough). ■

Proof of (17). Let $a, b \in H^1(0,\pi)$ and consider $\alpha := c^2 a(\psi)/\rho(\psi)$, $\beta := c^2 b(\psi)/\rho(\psi)$ constructed as above via the Liouville change of variable (64). By [20], p.34, for every j

$$|\lambda_j(a) - \lambda_j(b)| = \frac{1}{c^2}|\nu_j(\alpha) - \nu_j(\beta)| \leq \frac{1}{c^2}\|\alpha - \beta\|_\infty \leq K\|a - b\|_{H^1} \quad (68)$$

and (17) follows by the mean value theorem because $\mu\Pi_V f(v(\mu,w)+w)$ has bounded derivatives on bounded sets. ■

Proof of (47). By the asymptotic formula (61)

$$\min_{j\geq 1} |\omega_{j+1}(\mu,w) - \omega_j(\mu,w)| \geq \frac{1}{c} - \frac{K}{j} > \frac{1}{2c}$$

if $j > K/2c$, uniformly in $\mu \in [\mu_1,\mu_2]$, $w \in B_R$. For $1 \leq j \leq K/2c$ the minimum

$$m_j := \min_{(\mu,w)\in[\mu_1,\mu_2]\times B_R} |\omega_{j+1}(\mu,w) - \omega_j(\mu,w)|$$

is attained because $a \mapsto \lambda_j(a)$ is a compact function on H^1 by (68) and the compact embedding $H^1(0,\pi) \hookrightarrow L^\infty(0,\pi)$ (see also [20], Theorem 3 p.31 and p.34). Each $m_j > 0$ because all the eigenvalues λ_j are simple. ■

References

[1] P. Acquistapace, *Soluzioni periodiche di un'equazione iperbolica non lineare*, Boll. Un. Mat. Ital. B (5) **13** (1976), n.3, 760–777.

[2] A. Bamberger, G. Chavent, P. Lailly, *About the stability of the inverse problem in 1-D wave equations — applications to the interpretation of seismic profiles*, Appl. Math. Optim. **5** (1979), no.1, 1–47.

[3] V. Barbu, N.H. Pavel, *Periodic solutions to nonlinear one dimensional wave equation with x-dependent coefficients*, Trans. Amer. Math. Soc. **349** (1997), no.5, 2035–2048.

[4] T. Bartsch, Y.H. Ding, C. Lee, *Periodic solutions of a wave equation with concave and convex nonlinearities*, J. Differential Equations **153** (1999), no.1, 121–141.

[5] J. Berkovits, J. Mawhin, *Diophantine approximation, Bessel functions and radially symmetric periodic solutions of semilinear wave equations in a ball*, Trans. Amer. Math. Soc. **353** (2001), no. 12, 5041–5055.

[6] M. Berti, L. Biasco, *Forced vibrations of wave equations with non-monotone nonlinearities*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, available online 19 january 2006.

[7] M. Berti, P. Bolle, *Cantor families of periodic solutions for completely resonant nonlinear wave equations*, to appear in Duke Math. J.

[8] M. Berti, P. Bolle, *Cantor families of periodic solutions for wave equations via a variational principle*, preprint 2006.

[9] M. Berti, M. Procesi, *Quasi-periodic solutions of completely resonant forced wave equations*, Comm. Partial Differential Equations **31** (2006), 959–985.

[10] J. Bourgain, *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, Internat. Math. Res. Notices, no. 11, 1994.

[11] J. Bourgain, *Periodic solutions of nonlinear wave equations*, Harmonic analysis and partial differential equations, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.

[12] H. Brézis, L. Nirenberg: *Forced vibrations for a nonlinear wave equation*, Comm. Pure Appl. Math. **31** (1978), no. 1, 1–30.

[13] R. Courant, D. Hilbert, *Methods of mathematical physics, Vol. I.* Interscience Publishers, Inc., New York, 1953.

[14] W. Craig, *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et Synthèses, 9, Société Mathématique de France, Paris, 2000.

[15] W. Craig, E. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **46** (1993), 1409–1498.

[16] R. De La Llave, *Variational Methods for quasi-periodic solutions of partial differential equations*, World Sci. Monogr. Ser. Math. 6, 214–228, 2000.

[17] J.-M. Fokam, *Forced vibrations via Nash-Moser iteration*, preprint, `http://www.ma.utexas.edu/users/jmfokam`.

[18] P.J. McKenna, *On solutions of a nonlinear wave question when the ratio of the period to the length of the intervals is irrational*, Proc. Amer. Math. Soc. **93** (1985), no. 1, 59–64.

[19] P.I. Plotnikov, L.N. Yungerman, *Periodic solutions of a weakly nonlinear wave equation with an irrational relation of period to interval length*, translation in Diff. Equations **24** (1988), no. 9, 1059–1065 (1989).

[20] J. Pöschel, E. Trubowitz, *Inverse Spectral Theory*, Academic Press, Inc., Orlando, Florida, 1987.

[21] P. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math. **20** (1967), 145–205.

[22] P. Rabinowitz, *Time periodic solutions of nonlinear wave equations*, Manuscripta Math. **5** (1971), 165–194.

[23] P. Rabinowitz, *Free vibration of a semilinear wave equation*, Comm. Pure Appl. Math **31** (1978), no.1, 31–68.

[24] I.A. Rudakov, *Periodic solutions of a nonlinear wave equation with non-constant coefficients*, Math. Notes **76** (2004), no. 3, 395–406.

[25] W.M. Schmidt, *Diophantine Approximation*, Lect. Notes Math., v. 785, Springer Verlag, Berlin, 1980.

[26] K. Tanaka, *Infinitely many periodic solutions for the equation:* $u_{tt} - u_{xx} \pm |u|^{p-1}u = f(t,x)$, Trans. Am. Math. Soc. **307** (1988), no.2, 615–645.